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Free vibrational analysis of composite beams reinforced with randomly aligned and oriented carbon nanotubes, resting on an elastic foundation

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Abstract. The main interest of this paperwork is to examinate the dynamic behavior (free vibrational response) of carbon nanotubes (CNT) composite beams standing on an elastic foundation of Winkler-Pasternak's. The affected beam consists of a polymer matrix reinforced with single-wall carbon nanotubes (SWCNT's), in which, a large number of CNT's reinforcement of infinite length are distributed in a linear elastic polymer matrix. In this study the CNT's are considered either aligned or randomly oriented on the matrix.
A refined high-order beam theory (RBT) is adopted in the present analysis using a new shape function. The refined beam theory which is summarized by differentiating the displacement along the beam transverse section into shear and bending components, initially the material properties of the composite beam (CNTRC) are estimated using the Mori-Tanaka's method. The beam is considered simply supported on the edge-lines. NAVIER's solutions are proposed to solve the boundary conditions problems. Since there are no results to compare with in the literature; the results in this study are compared with a free vibrational analysis of an isotropic beam. Several aspects such as the length/thickness ratio, volume fraction of nanotubes, and vibrational modes are carried out in the parametric study.

Key words: Free vibration analyses, Mori-Tanaka's method, Carbon nanotube reinforced beams, Elastic foundation, refined beam theory.

1. Introduction

In the last few decades, carbon nanotubes (CNT's) were presented as a huge revelation in all construction fields because of their significant mechanical and electrical properties. CNT's were classified among the toughest materials in the world. In addition CNT's are easily employed as a result of their high flexibility. As researches continued to investigate, CNT's were becoming more usable especially in providing high performance materials for construction domains. Therefore, CNT's can be potentially integrated in the aerospace industry. (Thostenson et al., 2001; Esawi and Farag, 2007).

In civil engineering the preferable application of polymers/carbon nanotube is found in reinforcing structural elements such as beams and plates to improve several mechanical, thermal and electrical material characteristics. Furthermore, CNT's have been recently accepted as an excellent candidate for strengthening polymer composites because of their high elastic modulus, tensile strength and their low density which makes the resultant composites more efficient and remarkably light weighted.

The material properties of composites reinforced with carbon nanotubes (CNTRC) have been examined by many investigators, such as Fidelus et al. (2005) and Hu et al. (2005). In the same way, Shi et al. (2004) studied the stiffening effect of carbon nanotubes by employing the Mori-Tanaka effective-field method to calculate the effective elastic moduli of composites while considering the effects of waviness and agglomeration of CNT's on the effective stiffness.

On the other hand, there is still a lack of studies on the mechanical behavior of CNTRCs in the open literature. For example, Ke et al. (2010) analysed the non-linear free vibration of CNTRC using Timoshenko's theory of beams. Yas and Heshmati (2012) presented the dynamic response of nano composite beams with carbon nanotubes oriented randomly under a dynamic load. Wattanasakulpong and Ungbhakorn (2013) studied the bending, buckling and vibration behaviors of carbon nanotube-reinforced composite beams resting on elastic foundation. Furthermore, Tegrara et al. (2015) analyzed the mechanical behavior of nanotube-reinforced composite beams using the refined beam theory (RBT). Yas and Samadi (2012) evaluated the free vibrations and buckling responses of carbon nanotube-reinforced composite Timoshenko beams resting on elastic foundation.

In the current analysis, and in order to estimate the engineering constants (Young's modulus and Poisson's ratio) of composites with aligned or randomly oriented straight single-walled nanotubes in polymer matrix, Mori-Tanaka effective-field method is employed (Suresh, 1998). Thereafter, we aim to analyze the free vibrational response of CNT reinforced beams placed on elastic foundation.

2. Mathematical formulation

2.1. Material properties of composites reinforced with aligned CNT's

We consider first a polymer isotropic matrix with Young's modulus E_m, and Poisson's ratio ϑ_m. The polymer matrix is strengthened with straight transversely isotropic CNT's aligned in the x-axis direction (Figure 1). The stress-strain relation of the composite can be expressed as follow.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} k+m & l & k-m & 0 & 0 & 0 \\ l & n & l & 0 & 0 & 0 \\ k-m & l & k+m & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & p \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{sy} \end{Bmatrix} \quad (1)$$

Where k, m, l, n, and p are Hill's elastic moduli (Hill , 1965).

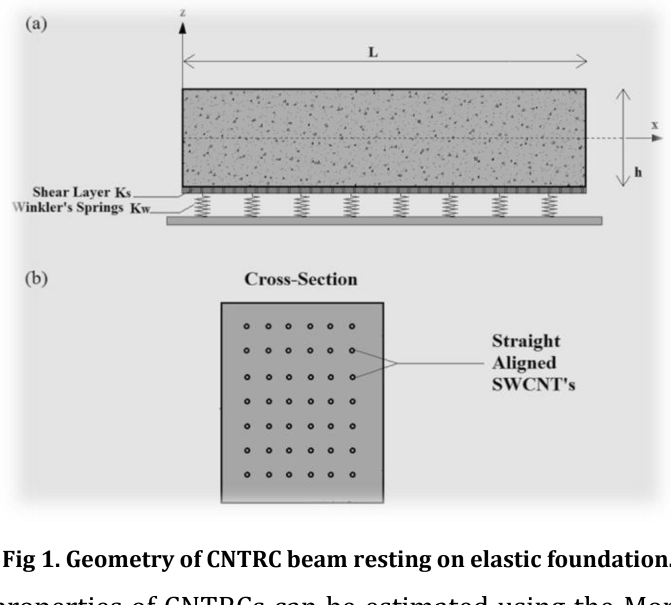

Fig 1. Geometry of CNTRC beam resting on elastic foundation.

The effective material properties of CNTRCs can be estimated using the Mori-Tanaka's method, such that:

(2)

$$k=\frac{E_m\left(E_mC_m+2k_r(1+\vartheta_m)\big(1+C_r(1-2\vartheta_m)\big)\right)}{2(1+\vartheta_m)\big(E_m(1+C_r-2\vartheta_m)+2C_rk_r(-2\vartheta_m^2-\vartheta_m+1)\big)}$$

$$l=\frac{E_m\left(C_m\vartheta_m+\big(E_m+2k_r(1+\vartheta_m)\big)+2C_rl_r(-\vartheta^2+1)\right)}{(1+\vartheta_m)\big(E_m(1+C_r-2\vartheta_m)+2C_mk_m(-2\vartheta_m^2-\vartheta_m+1)\big)}$$

$$n=\frac{E_m^2C_m\big((-C_m\vartheta_m+C_r+1)+2C_mC_r(k_rnr-ln^2)(1+\vartheta)^2(1-2\vartheta)\big)}{(1+\vartheta_m)\big(E_m(1+C_r-2\vartheta_m)+2C_mk_r(-2\vartheta_m^2-\vartheta_m+1)\big)}+\frac{E_m(2C_m^2k_r(1-\vartheta_m)+C_rn_r(1+C_r-2\vartheta_m)-4C_ml_r\vartheta_m)}{E_m(1+C_r-2\vartheta_m)+2C_mk_r(-2\vartheta_m^2-\vartheta_m+1)}$$

$$k=\frac{E_m\big(E_mC_m+2(1+C_r)p_r(1+\vartheta_m)\big)}{2(1+\vartheta_m)\big(E_m(1+C_r)+2C_mp_r(1+\vartheta_m)\big)}$$

$$k=\frac{E_m\big(E_mC_m+2m_r(1+\vartheta_m)(3+C_r-4\vartheta_m)\big)}{2(1+\vartheta_m)\left(E_m\big(C_m+4C_r(1-\vartheta_m)\big)+2C_mm_r(-4\vartheta_m^2-\vartheta_m+3)\right)}$$

Wherek_r, C_m, l_r, C_r, m_r, p_r, and n_r, are the elastic constants of SWCNT's.

Therefore, the expressions of the effective parallel and normal Young's modulus of CNTRCs are as follows.

$$E_{\parallel}=n-\frac{l^2}{k}$$
$$E_{\perp}=\frac{4m(kn-l^2)}{kn-l^2+mn} \quad (3)$$

2.2. Material properties of composites reinforced with randomly oriented CNT's

When CNTs are completely randomly oriented in the isotropic matrix with Young's modulus E_m, and poisson's ratio ϑ_m, the composite is then considered isotropic, and its bulk modulus K and shear modulus G are defined as:

$$K=K_m+\frac{C_r(-3k_m\alpha_r+\theta_r)}{3C_r\alpha_r+3C_m}$$
$$G=G_m+\frac{C_r(-2G_m\beta_r+\delta_r)}{2C_r\beta_r+2C_m} \quad (4)$$

Where

$$\alpha_r=\frac{3G_mk_m+k_r-l_r}{3G_m+k_r}$$
$$\theta_r=\frac{1}{3}\left(n_r+2l_r+\frac{(2k_r+l_r)(3k_m+2G_m-l_r)}{G_m+k_r}\right) \quad (5)$$

$$\delta_r=\frac{1}{15}(2n_r+2l_r)+\frac{1}{5}\left(\frac{8G_mp_r}{G_m+p_r}+\frac{8\big(m_rG_m(3k_m+4G_m)\big)}{3k_m(m_r+G_m)+G_m(7m_r+G_m)}\right) \quad (5)$$

$$\beta_r = \frac{1}{5}\left(\frac{4G_m + 2k_r + l_r}{3G_m + 3k_r} + \frac{4G_m}{G_m + p_r} + \frac{2\big(G_m(3k_m + G_m) + G_m(3k_m + 7G_m)\big)}{G_m(3k_m + G_m) + m_r(3k_m + 7G_m)}\right)$$

In which K_m, and G_m are the bulk and shear modulus of the polymer matrix respectively.

$$k_m = \frac{E_m}{3 - 6\vartheta_m}$$
$$G_m = \frac{E_m}{2 + 2G_m} \tag{6}$$

The effective Young's modulus E and Poisson's ratio n of the composite are given by:

$$E = \frac{9KG}{3K + G}$$
$$\vartheta = \frac{3K - 2G}{6K + 2G} \tag{7}$$

2.3. Displacement field

Based on the refined plate theory assumptions (Shimpi et al., 2006), the displacement field in the refined theory can be written as:

$$\begin{cases} U(x, y, z) = u_0(x, y) - z\dfrac{dw_b(x, y)}{dx} - f(z)\dfrac{dw_s(x, y)}{dx} \\ W(x, y, z) = w_b(x, y) + w_s(x, y) \end{cases} \tag{8}$$

Where u_0 is the mid-plane displacement of the beam along the x-direction, 'w_b' and 'w_s' are the bending and shear components of transverse displacement in z-direction, respectively.

While the function f(z) represents shape functions determining the distribution of the transverse shear strains and stresses across the plate thickness; if the function is neglected, the displacements are reduced to the classical plate theory (CPT), else if the function is linear, the displacements are reduced to the first order deformation theory (FSDT).

In this analysis a new shape functions are proposed.

$$f(z) = 2\pi\left(\frac{z}{h}\right)^3 - \frac{3\pi}{4h}z \tag{9}$$

It should be noted that unlike the first-order shear deformation theory, these theories do not require shear correction factors. The linear strain expressions associated with the displacements in the equation 8, are:

$$\varepsilon_x = \varepsilon_x^0 - zk_x^b + f(z)k_x^s$$
$$\gamma_{xz} = \left(1 - \frac{df(z)}{dz}\right)\gamma_{xz}^s = g(z)\gamma_{xz}^s \tag{10}$$

The strain components derived from the displacement field are well founded for thin and thick plates, where:

$$\varepsilon_x^0 = \frac{dU}{dx}; k_x^b = \frac{d^2w_b}{dx^2}; k_x^s = \frac{d^2w_s}{dx^2}; \gamma_{xz}^s = \frac{dw_s}{dx}. \tag{11}$$

In which, the prime indicates differentiation of the function with respect to z, such that:

$$f'(z) = \frac{df(z)}{dz},\ \ g(z) = 1 - \frac{df(z)}{dz}. \tag{12}$$

The expression of the constitutive relations can be expressed as:

$$\begin{aligned} \sigma_x &= Q_{11}\varepsilon_x \\ \tau_{xz} &= Q_{55}\gamma_{xz} \end{aligned} \tag{13}$$

Where Q_{ij} are the elastic constants, namely.

$$Q_{11} = \frac{E_{\parallel}}{1 - \vartheta_{12}\vartheta_{21}}, Q_{55} = G_{13}. \tag{14}$$

2.4. Governing equations

The virtual work's principle is applied to develop the equations of motion:

$$\int_0^t (\delta U + \delta V - \delta K)\, dt = 0 \tag{15}$$

Where δU and δV are the virtual variation of the internal strain energy, the virtual work done by external forces.

Firstly, the expression of the virtual strain energy is.

$$\delta U = \sum_{n=1}^{N} \int_{h_n}^{h_{n+1}} \int_A \ \sigma_x \delta\varepsilon_{xx} + \sigma_y \delta\varepsilon_{yy} + \sigma_{xy}\delta\gamma_{xy} + \sigma_{yz}\delta\gamma_{yz} + \sigma_{xz}\delta\gamma_{xz} \Big) dAdx \tag{16}$$

By substituting equation 10 into equation 16, we find:

$$\delta U = \int_A \{N_x \delta u_{0,x} - M_x^b \delta w_{b,x} + M_x^s \delta w_{s,x} + Q_{xz}(w_{s,x}) + Q_{yz}(w_{s,y})\} dxdy \tag{17}$$

By substituting equation 14 into equation 17, we obtain the stress resultants in form of material stiffness and displacement components:

$$N_x = A\frac{\partial u_0}{\partial x} + B\frac{\partial^2 w_b}{\partial x^2} + B_s\frac{\partial^2 w_s}{\partial x^2} \tag{18a}$$

$$M_x^b = B\frac{\partial u_0}{\partial x} + D\frac{\partial^2 w_b}{\partial x^2} + D_s\frac{\partial^2 w_s}{\partial x^2} \tag{18b}$$

$$M_x^s = B_s\frac{\partial u_0}{\partial x} + D_s\frac{\partial^2 w_b}{\partial x^2} + H_s\frac{\partial^2 w_s}{\partial x^2} \tag{18c}$$

$$\begin{Bmatrix} Q_{yz} \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} As_{44} & 0 \\ 0 & As_{55} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w_s}{\partial y} \\ \dfrac{\partial w_s}{\partial x} \end{Bmatrix} \tag{18d}$$

Where A, B, D, Ds, Bs, Hs, As_{ij}, are the plate stiffness, defined by:

$$[A_{ij}, B_{ij}, D_{ij}] = \sum \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}[1, z, z^2]\, dz$$

$$[Bs_{ij}, Ds_{ij}, Hs_{ij}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}[\mathrm{f(z)}, zf(z), f(z)^2]\, dz \tag{19}$$

$$[As_{ij}] = \sum_{n=1}^{N} \int_{h_n}^{h_{n+1}} C_{ij} g(z)^2\, dz\,; \qquad i,j = 4,5$$

The expression of the virtual work done by external loads while considering the effect of the elastic foundation can be expressed as follow.

$$\delta V = -\int_A q\delta w_0\, dxdy + \int_0^L \left(K_w (w_b + w_s)(\delta w_b + \delta w_s) + K_s \frac{d(w_b + w_s)}{dx} \frac{d(\delta w_b + \delta w_s)}{dx} \right) \tag{20}$$

Where Kw, and Ks are the Winkler and shearing layer spring constants.

For the dynamic analysis, the virtual kinetic energy (δK) is required for the equations of motion, which takes the form

$$\delta K = \int_0^L \rho(z)\left(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}\right) \tag{21}$$

Following the NAVIER closed-form solutions, we assume the following solution form for the displacement functions expanded in double trigonometric Fourier's series that satisfies the boundary conditions.
At edges $x = 0$ and $x = a$

Either $N_x = 0$ or u_0 is prescribed

Either $M_x^b = 0$ or dw_b / dx is prescribed

Either $M_x^s = 0$ or is prescribed

Where stress resultants can be expressed as follows:

$$\left(\mathrm{N_x, M_x^b, M_x^s, Q_{xz}, Q_{yz}}\right) = \sum \int_{-\frac{\mathrm{h}}{2}}^{\frac{\mathrm{h}}{2}} \left(\sigma_x, z\sigma_x, f(z)\sigma_x, g(z)\sigma_{xz}, g(z)\sigma_{yz}\right) \mathrm{dz} \tag{22a}$$

and I_0, I_2 are mass inertias defined as:

$$(\mathrm{I_0, I_2}) = \sum \int_{\mathrm{A}} \rho(\mathrm{z})(1, \mathrm{z}^2)\, \mathrm{dA} \tag{22b}$$

By substituting equation 8 into equation 13, we obtain the stress resultants in form of material stiffness and displacement components:

$$\delta u_0 \colon \frac{\partial N_x}{\partial x} = I_0 \ddot{u}_0$$

$$\delta w_b : \frac{\partial^2 M_x^b}{\partial x^2} - K_w(w_b + w_s) + K_s\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2}\right) = I_0(\ddot{w}_b + \ddot{w}_s) - I_2\frac{d^2\ddot{w}_b}{dx^2}$$

$$\delta \mathrm{w_b} : \frac{\partial^2 \mathrm{M_x^s}}{\partial \mathrm{x}^2} + \frac{\partial \mathrm{Q_{xz}}}{\partial \mathrm{x}} + \frac{\partial \mathrm{Q_{yz}}}{\partial \mathrm{y}} - K_w(w_b + w_s) + K_s\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2}\right) = I_0(\ddot{w}_b + \ddot{w}_s) - \frac{I_2}{84}\frac{d^2\ddot{w}_s}{dx^2}$$

2.5. NAVIER solutions

To formulate the closed-form solutions for bending and buckling problems of simply supported laminated plates, the NAVIER method is employed:

$$u_0(x,y,t) = \sum_{n=1}^{\infty} U_{mn}\cos(\alpha x)$$
$$w_b(x,y,t) = \sum_{N=1}^{\infty} W_{mn}^b \sin(\alpha x) \tag{23}$$
$$w_s(x,y,t) = \sum_{n=1}^{\infty} W_{mn}^s \sin(\alpha x)$$

WhereU_{mn} ,W_{bmn} and $W_{smn}$are the arbitrary parameters to be determined. $\alpha = m\pi/a$, and n are vibrational mode shape.

Substituting equation 23 into the equilibrium equations, we obtain the closed-form solutions which are presented in the following matrix form.

$$\left(\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\right) = 0 \tag{24}$$

Where

$$s_{11} = -A_{11}\alpha^2,\ s_{12} = B\alpha^3,\ s_{13} = Bs\alpha^3,$$
$$s_{21} = s_{12},\ s_{22} = -D\alpha^4, s_{23} = -Ds\alpha^4,$$
$$s_{31} = s_{13},\ s_{32} = s_{23},\ s_{33} = -Hs\alpha^4 - As\alpha^2.$$

$$m_{11} = m_{23} = I_0, m_{22} = I_0 + I_2\alpha^2, m_{33} = I_0 + \frac{I_2}{84}\alpha^2.$$

3. Results and Discussions

The NAVIER solution was employed to determine the natural frequencies of CNT composite beams by solving the eigenvalue (equation 24). Before analyzing the free vibrations of carbon nanotubes reinforced composite (CNTRC) beams resting on Winkler-Pasternak elastic foundation, the material properties were calculated and presented in (Figure 2) for the aligned CNT's and (Figure 3) oriented CNT's, these properties (Young's modulus) were defined using the Mori-Tanaka's approach, such that the Young's modulus and Poisson's ratio of polystyrene are $Em = 1.9\ GPa$ and $\vartheta m = 0.3$, respectively. For the reinforcement, we use the following representative values of the elastic constants of SWCNT's: $nr = 450\ GPa, kr = 30\ GPa, pr = mr = 1\ GPa$, and $lr = 10\ GPa$, which are taken from the analytical results of Popov et al. (2000). In which kr, mr, nr, pr and lr are the Hill's elastic moduli for the reinforcing phase (CNT's).

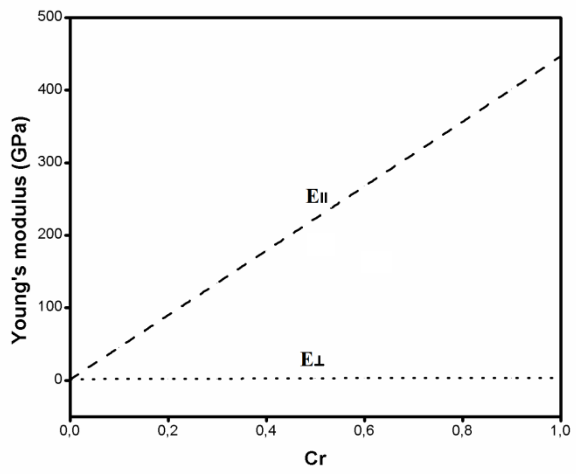

Fig 2. Young's modulus in terms of the fraction volume of aligned CNT's

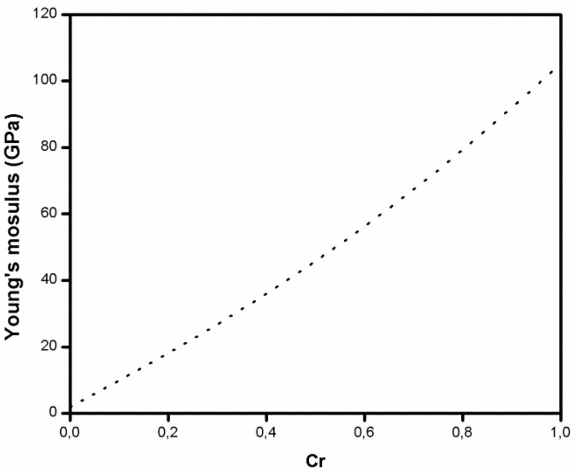

Fig 3. Young's modulus in terms of the fraction volume of oriented CNT's

Figures 2 and 3 show the variation of young's modulus of CNTC's in terms of the volume fraction of CNT's, knowing that the young modulus of CNT's in the fibers direction is two orders of magnitude higher than the normal young modulus, the CNT's are considered highly anisotropic. It is observed from (Figure 2) that, because of CNTs' anisotropic property, the elastic modulus of the composite in the reinforcement direction increases much more rapidly with the volume fraction "cr" than the normal to the CNT direction. When the CNT's volume fraction cr=0, the composite is pure isotropic polystyrene. In a similar way, (Figure 3) presents the effective Young's modulus versus the volume fraction of randomly oriented, straight CNTs in the same polystyrene matrix, it shows that the young modulus of the oriented carbon nanotubes reinforcement increases in parallel with the increase of the volume fraction of CNT's.

From Figures 2 and 3 it can be seen that the aligned CNT's in the polystyrene matrix is much more effective than the oriented CNT's in terms of the young modulus magnitude.

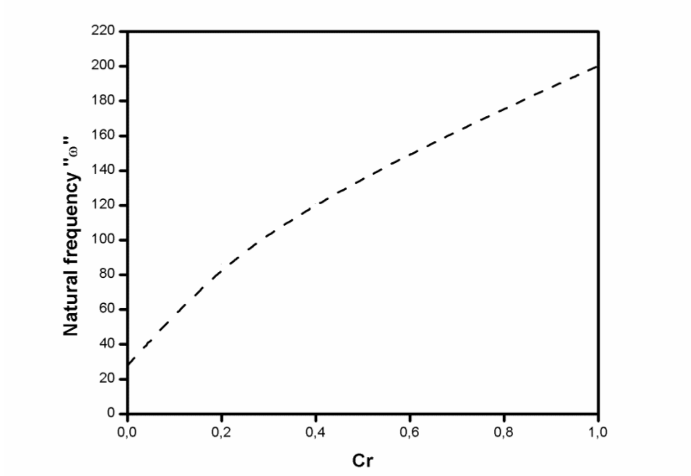

Fig 4. Dimensionless natural frequency of CNTRC beam (L=10h, n=1).

For the vibrational analysis of CNTRC beams resting of elastic foundation, and in order to verify the accuracy of the present mathematical models and the proposed shape shear function in predicting vibrational analysis of beams. We used the following properties: $Em = 1.9\, GPa$, $\vartheta m = 0.3$ and $\rho m = 1190\, kg/m^3$ for the polymer matrix. $nr = 450\, GPa$, $kr = 30\, GPa$, $pr = mr = 1\, GPa$, $lr = 10\, GPa$, and $cr = open$, for the SWCNT's reinforcements.

All analytical results are presented in the dimensionless forms which can be written as follows:

$$\omega = wL\sqrt{\frac{I_{00}}{A_{10}}}$$

Where A_{10} and I_{00} are A and I_0 of beam made of pure matrix material, respectively.

For the elastic foundation spring constants, the following expressions are used:

$$K_w = \beta_w \frac{A_{10}}{L^2} \qquad K_s = \beta_s A_{10}$$

Figures 4 and 5, present the dimensionless frequencies of CNTRC beam with, the reinforcement which are considered oriented in the polystyrene matrix, the influence of CNT volume fraction is obvious in compare to an isotropic polymer beam (Cr=0), the more "cr" presence gets raised in the matrix, more the dimensionless natural frequency increased.

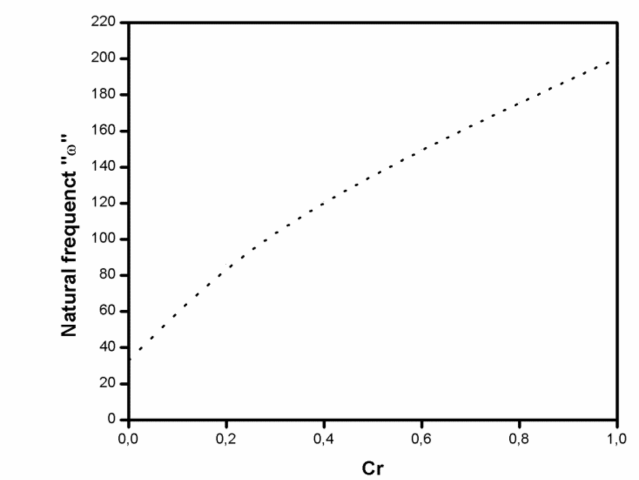

Fig 5. Dimensionless natural frequency of CNTRC beam (L=10h, n=1, $\beta_w = 100, \beta_s = 100$).

4. Conclusions

In the present study, the material properties of carbon nanotubes reinforced composite beams are defined using the Mori-Tanaka's method, while considering aligned and randomly oriented CNT's, it is concluded that the aligned reinforcement in the polymer matrix is much more effective than the randomly oriented CNT's, because CNT's laid in an aligned way have high properties due to the high elastic properties of the CNT's in disposition direction.

As well a dynamic study of CNTRC beams was presented in this work with and without the elastic foundation, the spring β_w and shear layer β_s constants of the elastic foundation have a very minor effect on the vibration frequencies regardless of CNT's volume fraction in the polymer matrix.

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